

BLUE SCHEMES AS RELATIVE SCHEMES AFTER TOËN AND VAQUIÉ

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ABSTRACT. In this note, we show that blue schemes over a base blueprint B are naturally schemes relative to blue B -modules (in the sense of Toën and Vaquié). However, there are relative schemes that do not come from blue schemes. This discrepancy occurs already for semiring schemes and schemes relative to \mathbb{N} -modules.

INTRODUCTION

Bertrand Toën and Michel Vaquié developed in [6] the notion of a scheme relative to any complete and cocomplete closed symmetric monoidal category \mathcal{C} . One of the key ingredients of this vast generalization of the machinery of scheme theory is the following.

The usual definition of open subsets is in terms of localizations of a ring R at elements $h \in R$. This can be generalized to other structures than rings (e.g. monoids or blueprints), but it requires the notion of an underlying set. Toën and Vaquié use the categorical viewpoint as developed in Demazure and Gabriel's book [3]. Namely, a localization is the same as a flat epimorphism of finite presentation, which allows a completely categorical characterization of open immersions. This leads to the notion of a Zariski site on the (dual of) the category of monoids in \mathcal{C} . Roughly said, a scheme relative to \mathcal{C} is a sheaf on this site that admits a covering by affine opens.

In case, \mathcal{C} is the category of R -modules, this reproduces the category of schemes over a ring R . Alberto Vezzani shows in [7] that if \mathcal{C} is the category of sets, which has the interpretation as the category of \mathbb{F}_1 -modules, then schemes relative to \mathcal{C} can be identified with \mathbb{F}_1 -schemes in the sense of Deitmar ([2]).

This text is concerned with the category of blue schemes, which includes both usual schemes and Deitmar's \mathbb{F}_1 -schemes as full subcategories. The main result Theorem 7 of this paper is that the category of blue schemes over a base blueprint B embeds as a full subcategory into the category of scheme relative to the category $\mathcal{M}od B$ of blue B -modules.

In the category of blueprints, it happens that not every flat epimorphism $B \rightarrow C$ of finite presentation is a localization of B at an element $h \in B$. This means that there are more open immersions in the sense of Toën and Vaquié than can be seen algebraically. Indeed, there are relative schemes that are not blue schemes. Interesting enough for applications to tropical geometry or analytic geometry, which are based on semirings, this discrepancy already occurs for semiring schemes. Another curiosity is that all affine blue schemes are spectra of global blueprints while relative schemes also include affine schemes that are "not global".

This text is organized as follows. After recalling the definitions of relative schemes (Section 1), blueprints (Section 2) and blue modules (Section 3), we show in Section 4 that monoids in the category of blue B -modules coincide with B -algebras. In Section 5, we characterize morphisms of finite presentation in terms of algebraic properties of blueprints. After reviewing the construction of localizations (Section 6), we conclude in Section 7 that localizations are flat epimorphisms of finite presentation and state the main theorem. In Section 8, we give an example of a relative scheme that is not a blue scheme. We finish the paper with a remark on globalizations in Section 9.

Note that we provide only those details in the proofs of this short text that are essentially different from arguments in similar contexts. If the techniques of proof occur already at other places, then we restrict ourselves to a reference to don't repeat reasonings that require

a heavily technical language. In particular, we omit general sheaf theoretic arguments, which are explained very clearly in [7].

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1. RELATIVE SCHEMES AFTER TOËN AND VAQUIÉ

We recall the definition of relative schemes from Toën and Vaquié's paper [6]. Let \mathcal{C} be a closed symmetric monoidal category that is complete and cocomplete. We denote by $\text{Comm}(\mathcal{C})$ the category of commutative, associative and unital monoids in \mathcal{C} . A *B-algebra* is a morphism $B \rightarrow C$ of commutative monoids in \mathcal{C} and a *B-algebra morphism* is a morphism $C \rightarrow C'$ that commutes with the morphisms $B \rightarrow C$ and $B \rightarrow C'$.

Let $f : B \rightarrow C$ be a morphism of commutative monoids in \mathcal{C} . The morphism f is *flat* if $-\otimes_B C$ commutes with finite limits and colimits. The morphism $f : B \rightarrow C$ is of *finite representation* if for all directed systems \mathcal{D} of B -algebras D_i together with B -algebra morphisms $\varphi_{i,j} : D_i \rightarrow D_j$, the canonical map

$$\Psi_{\mathcal{D}} : \text{colim } \text{Hom}_B(C, \mathcal{D}) \longrightarrow \text{Hom}_B(C, \text{colim } \mathcal{D})$$

is bijective.

An *affine scheme relative to \mathcal{C}* is an object of the dual category $\text{Comm}(\mathcal{C})$, which we denote by $\text{Aff}(\mathcal{C})$. Let $\text{spec} : \text{Comm}(\mathcal{C}) \rightarrow \text{Aff}(\mathcal{C})$ be the contravariant equivalence of dual categories, which sends a morphism f to f^* . Let $f : B \rightarrow C$ be a morphism of commutative monoids in \mathcal{C} . Then $f^* : \text{spec } C \rightarrow \text{spec } B$ is called a *Zariski open immersion* if $f : B \rightarrow C$ is a flat epimorphism of finite representation.

The category $\text{Comm}(\mathcal{C})$ of affine \mathcal{C} -schemes admits a notion of Zariski coverings that is based on Zariski open immersions. This endows $\text{Comm}(\mathcal{C})$ with the structure of a site, and it makes sense to consider sheaves on $\text{Comm}(\mathcal{C})$. The functor $h_{\text{spec } B} = \text{Hom}(-, \text{spec } B)$ of points is a sheaf of sets on $\text{Aff}(\mathcal{C})$. The notion of Zariski coverings extends to sheaves on $\text{Aff}(\mathcal{C})$ in a natural way. A *scheme relative to \mathcal{C}* is a sheaf on $\text{Aff}(\mathcal{C})$ that has a Zariski covering by affine open subschemes. If the category \mathcal{C} is understood, we sometimes refer to a scheme relative to \mathcal{C} simply by a *relative scheme*. See [6] for more details on the definition of relative schemes.

2. BLUEPRINTS

By a *monoid with zero*, we mean in this text always a multiplicatively written commutative semigroup A with a neutral element 1 and an absorbing element 0 , which are characterized by the properties $1 \cdot a = a$ and $0 \cdot a = 0$ for all $a \in A$. A *morphism of monoids with zero* is a multiplicative map $f : A_1 \rightarrow A_2$ that maps 1 to 1 and 0 to 0 . We denote the category of monoids with zero by \mathcal{M}_0 .

A *blueprint* B is a monoid A with zero together with a *pre-addition* \mathcal{R} , i.e. \mathcal{R} is an equivalence relation on the semiring $\mathbb{N}[A] = \{\sum a_i | a_i \in A\}$ of finite formal sums of elements of A that satisfies the following axioms (where we write $\sum a_i \equiv \sum b_j$ whenever $(\sum a_i, \sum b_j) \in \mathcal{R}$):

- (i) $\sum a_i \equiv \sum b_j$ and $\sum c_k \equiv \sum d_l$ implies $\sum a_i + \sum c_k \equiv \sum b_j + \sum d_l$ and $\sum a_i c_k \equiv \sum b_j d_l$,
- (ii) $0 \equiv (\text{empty sum})$, and
- (iii) if $a \equiv b$, then $a = b$ (as elements in A).

A *morphism $f : B_1 \rightarrow B_2$ of blueprints* is a multiplicative map $f : A_1 \rightarrow A_2$ between the underlying monoids of B_1 and B_2 , respectively, with $f(0) = 0$ and $f(1) = 1$ such that for every relation $\sum a_i \equiv \sum b_j$ in the pre-addition \mathcal{R}_1 of B_1 , the pre-addition \mathcal{R}_2 of B_2 contains the relation $\sum f(a_i) \equiv \sum f(b_j)$. Let \mathcal{Blpr} be the category of blueprints.

Remark 1. Note that the above definition follows the convention of the paper [5], which allows us to use an elegant and intuitive notation. In the sense of [4], the definition above

agrees with a *proper blueprint with zero* from [4]. Therefore the category of blue schemes as considered in this text does, strictly speaking, not contain the category of \mathbb{F}_1 -schemes in the sense of Deitmar ([2]). However, all arguments of [7] will transfer to show an equivalence of monoidal schemes *with zero*. Alternatively, one can extend the results of this text to the more general setting of blue schemes in [4], which includes \mathbb{F}_1 -schemes after Deitmar.

In the following, we write $B = A // \mathcal{R}$ for a blueprint B with underlying monoid A and pre-addition \mathcal{R} . We adopt the conventions used for rings: we identify B with the underlying monoid A and write $a \in B$ or $S \subset B$ when we mean $a \in A$ or $S \subset A$, respectively. Further, we think of a relation $\sum a_i \equiv \sum b_j$ as an equality that holds in B (without the elements $\sum a_i$ and $\sum b_j$ being defined, in general).

Given a set S of relations, there is a smallest equivalence relation \mathcal{R} on $\mathbb{N}[A]$ that contains S and satisfies Axioms (i) and (ii). If \mathcal{R} satisfies also Axiom (iii), then we say that \mathcal{R} is the pre-addition generated by S , and we write $\mathcal{R} = \langle S \rangle$. In particular, every monoid A with zero has a smallest pre-addition $\mathcal{R} = \langle \emptyset \rangle$.

More generally, let A be a monoid with zero and \mathcal{R} an equivalence relation on $\mathbb{N}[A] \times \mathbb{N}[A]$ that satisfies Axioms (i) and (ii). We can form the quotient set $A' = A / \sim$ where $a \sim b$ whenever $a \equiv b$. Then A' inherits the structure of a monoid by the multiplicativity of \mathcal{R} , and the image \mathcal{R}' of \mathcal{R} in $\mathbb{N}[A'] \times \mathbb{N}[A']$ is a pre-addition on A' , satisfying Axiom (iii) (see Lemma 1.6 in [4] for more details on the construction of the proper quotient). We say that $A // \mathcal{R}$ is a *representation* of the blueprint $A' // \mathcal{R}'$, and we say that the representation $A // \mathcal{R}$ of $B = A' // \mathcal{R}'$ is *proper* if $A = A'$.

It is possible to associate with a blueprint its *spectrum* $X = \text{Spec } B$, which is the set of prime ideals of B together with its Zariski topology, which is defined in terms of localizations (cf. Section 6), and together with a structure sheaf \mathcal{O}_X , which is a sheaf in blueprints. The pair (X, \mathcal{O}_X) forms a so-called *locally blueprinted space*. A *blue scheme* is a locally blueprinted space that is locally isomorphic to the spectrum of a blueprint. For more details on the definition of blue schemes, see [4].

3. BLUE B -MODULES

Let M be a pointed set. We denote the base point of M by $*$. A *pre-addition on M* is an equivalence relation \mathcal{P} on the semigroup $\mathbb{N}[M] = \{\sum a_i | a_i \in M\}$ of finite formal sums in M with the following properties (as usual, we write $\sum m_i \equiv \sum n_j$ if $\sum m_i$ stays in relation to $\sum n_j$):

- (i) $\sum m_i \equiv \sum n_j$ and $\sum p_k \equiv \sum q_l$ implies $\sum m_i + \sum p_k \equiv \sum n_j + \sum q_l$,
- (ii) $*$ \equiv (empty sum), and
- (iii) if $m \equiv n$, then $m = n$ (in M).

Let $B = A // \mathcal{R}$ be a blueprint. A *blue B -module* is a set M together with a pre-addition \mathcal{P} and a *B -action* $B \times M \rightarrow M$, which is a map $(b, m) \mapsto b.m$ that satisfies the following properties:

- (i) $1.m = m$, $0.m = *$ and $a.* = *$,
- (ii) $(ab).m = a.(b.m)$, and
- (iii) $\sum a_i \equiv \sum b_j$ and $\sum m_k \equiv \sum n_l$ implies $\sum a_i.m_k \equiv \sum b_j.n_l$.

A *morphism of blue B -modules* M and N is a map $f : M \rightarrow N$ such that

- (i) $f(a.m) = a.f(m)$ for all $a \in B$ and $m \in M$ and
- (ii) whenever $\sum m_i \equiv \sum n_j$ in M , then $\sum f(m_i) \equiv \sum f(n_j)$ in N .

This implies in particular that $f(*) = *$. We denote the category of blue B -modules by $\mathcal{Mod} B$.

Lemma 2. *The category $\mathcal{Mod} B$ is closed, complete and cocomplete. The trivial blue module $0 = \{*\}$ is an initial and terminal object of $\mathcal{Mod} B$.*

Proof. All arguments are essentially the same as in the case of A -sets. We refer to [1, Section 2.2.1] for the facts that $\mathcal{M}odB$ is closed and 0 is initial and terminal. The construction of limits and colimits can be found in [1, Prop. 2.13]. \square

Lemma 3. *The category $\mathcal{M}odB$ has tensor products $M \otimes_B N$, which are characterized by the universal property that every bi- B -linear morphism $M \times N \rightarrow P$ factors through a unique B -linear map $M \otimes_B N \rightarrow P$. The functor $- \otimes_B M$ is left-adjoint to $\text{Hom}_B(M, -)$. Together with the tensor product, $\mathcal{M}odB$ is a symmetric monoidal category.*

Proof. The construction of the tensor product is the same as in Section 2.2.3 of [1]. All further details are easily verified. \square

4. MONOIDS IN $\mathcal{M}odB$

Let B be a blueprint. We denote the category of B -algebras by $\mathcal{B}lpr_B$ and its morphism sets by $\text{Hom}_B(C, C')$. Let \mathbb{F}_1 be the blueprint $\{0, 1\} // \langle \emptyset \rangle$, which is an initial object in $\mathcal{B}lpr$. Then the association $(\mathbb{F}_1 \rightarrow B) \mapsto B$ establishes an equivalence between $\mathcal{B}lpr_{\mathbb{F}_1}$ and $\mathcal{B}lpr$.

Lemma 4. *Let B be a blueprint. Then the category $\mathcal{B}lpr_B$ is equivalent to the category of commutative monoids in $\mathcal{M}odB$.*

Proof. A B -algebra $f : B \rightarrow C$ is a blue B -module w.r.t. to the multiplication defined by $b.c = f(b)c$ for $b \in B$ and $c \in C$. The multiplication of C turns C into a commutative monoid in $\mathcal{M}odB$. A morphism $C \rightarrow C'$ of B -algebras induces naturally a morphism of the associated commutative monoids in $\mathcal{M}odB$. It is immediately verified that this functor is an equivalence of categories. \square

5. BLUEPRINT MORPHISMS OF FINITE PRESENTATION

In this section, we characterize of morphisms $f : B \rightarrow C$ of finite presentation in terms of the finiteness of certain sets of generators.

Let $B = A // \mathcal{R}$ be a blueprint. Then we denote by $B[T_i]_{i \in I}$ the free blueprint over B in the indeterminants T_i , cf. 1.12 of [4]. Its elements are $\{0\}$ and all monomials $b \prod T_i^{n_i}$ with coefficients $b \in B - \{0\}$ where $n_i \geq 0$ with $n_i = 0$ for almost but finitely many $i \in I$. The blueprint B can be seen as the subset of all constants $b \prod T_i^0$, and the pre-addition of $B[T_i]$ is generated by the image of the \mathcal{R} in $B[T_i]$.

A B -algebra $f : B \rightarrow C$ is *generated by a subset* $\{b_i\}_{i \in I}$ of C if there are for every element $c \in C$ finitely many $a_i \in B$ and not necessarily different indices in $i_i \in I$ such that $c \equiv \sum f(a_i)b_{i_i}$. A *presentation of a B -algebra* $f : B \rightarrow C$ is pair (\mathfrak{b}, S) where $\mathfrak{b} = \{b_i\}_{i \in I}$ generates $f : B \rightarrow C$ and S is a set of relations on the free B -algebra $B[T_i]_{i \in I}$ that satisfies the following property: for $\tilde{B} = B[T_1, \dots, T_n] // \langle S \rangle$, there is a monomorphism $\tilde{f} : \tilde{B} \rightarrow C$ of B -algebras that sends T_i to b_i and the pre-addition of C is generated by the image of the pre-addition of \tilde{B} . Note that the underlying monoid of \tilde{B} is in general a proper quotient of $B[T_1, \dots, T_n]$.

A B -algebra $f : B \rightarrow C$ is *algebraically of finite presentation* if there is a presentation (\mathfrak{b}, S) of $f : B \rightarrow C$ with finite \mathfrak{b} and finite S . We say that such a pair (\mathfrak{b}, S) is a *finite presentation* for $f : B \rightarrow C$.

Proposition 5. *A morphism $f : B \rightarrow C$ of blueprints is of finite presentation if and only if it is algebraically of finite presentation.*

Proof. We unfold the definition of a finitely presented morphism of blueprints. Consider a directed system \mathcal{D} of B -algebras D_i (where i ranges through an index set I) and morphisms $f_{i,j} : D_i \rightarrow D_j$ (for a directed subset of indices $(i, j) \in I \times I$). Define $J(i) = \{j \in I \mid \text{there is a } \varphi_{i,j} : D_i \rightarrow D_j\}$. Then the colimit of \mathcal{D} can be represented by the B -algebra

$$\text{colim } \mathcal{D} = \coprod_{i \in I} \left\{ (a_j) \in \prod_{j \in J(i)} D_j \mid \forall f_{i,j} : D_i \rightarrow D_j, a_k = f(a_j) \right\} / \sim$$

where two elements $(a_j)_{j \in J(i_1)}$ and $(b_j)_{j \in J(i_2)}$ are equivalent if $a_j = b_j$ for all $j \in J(i_0)$ for some $i_0 \in J(i_1) \cap J(i_2)$. The canonical morphism $\iota_i : D_i \rightarrow \text{colim} D_i$ maps $a_i \in D_i$ to $(f_{i,k}(a_i) \mid f_{i,k} : D_i \rightarrow D_k)$.

Similarly, an element of $\text{colim Hom}_B(C, \mathcal{D})$ can be represented as a tuple $(\varphi_j : B \rightarrow D_j)_{j \in J(i)}$ that satisfies $\varphi_k = f_{j,k} \circ \varphi_j$ for all $j, k \in J(i)$ where i is some element of I . The canonical map

$$\Psi_{\mathcal{D}} : \text{colim Hom}_B(C, \mathcal{D}) \longrightarrow \text{Hom}_B(C, \text{colim } \mathcal{D})$$

send the class of a tuple $(\varphi_j : B \rightarrow D_j)_{j \in J(i)}$ to the morphism $\varphi = \iota_i \circ \varphi_i : C \rightarrow \text{colim } \mathcal{D}$. Then $F : B \rightarrow C$ is of finite presentation if $\Psi_{\mathcal{D}}$ is a bijection for all directed systems \mathcal{D} .

Assume $(\{b_1, \dots, b_n\}, S)$ is a finite presentation for $f : B \rightarrow C$. Let \mathcal{D} be a directed system.

We show that $\Psi_{\mathcal{D}}$ is injective. Let $(\varphi_j)_{j \in J(i_1)}$ and $(\psi_j)_{j \in J(i_2)}$ be two elements of $\text{colim Hom}_B(C, D_i)$ such that $\varphi = \Psi_{\mathcal{D}}(\varphi_j) = \Psi_{\mathcal{D}}(\psi_j) = \psi$. This means that for every $l = 1, \dots, n$, there is an $j_l \in J(i_1) \cap J(i_2)$ such that $\varphi_{j_l}(b_l) = \psi_{j_l}(b_l)$ in D_{j_l} . If $j \in J(j_1) \cap \dots \cap J(j_n)$, then $\varphi_j(b_l) = \psi_j(b_l)$ in D_j for all $l = 1, \dots, n$. Therefore, we obtain for an arbitrary element $c \equiv \sum f(a_k) b_{l_k}$ in C , the relation

$$\varphi_j(c) \equiv \sum f(a_k) \varphi_j(b_{l_k}) \equiv \sum f(a_k) \psi_j(b_{l_k}) \equiv \psi_j(c),$$

i.e. $(\varphi_j) = (\psi_j)$. This shows the injectivity of $\Psi_{\mathcal{D}}$.

We show that $\Psi_{\mathcal{D}}$ is surjective. Let $\varphi : C \rightarrow \text{colim } \mathcal{D}$ be a morphism of B -algebras. Then for every relation R in $\tilde{f}(S)$, there is an $i_R \in I$ and $c_l \in D_{i_R}$ such that $\varphi(b_l) = \iota_{i_R}(c_l)$ for $l = 1, \dots, n$ and such that the c_l satisfy the relation $\varphi_{i_R}(R)$. Since S is finite, we can replace the i_R by an i in $\bigcup J(i_R)$ and can assume that there are elements c_l in D_i that satisfy all relations in $\varphi_i(\tilde{f}(S))$ and such that $\iota_i(c_l) = b_l$. This means that $\varphi : C \rightarrow \text{colim } \mathcal{D}$ factors into a morphism $\varphi_i : C \rightarrow D_i$, defined by $\varphi_i(b_l) = c_l$, followed by $\iota_i : D_i \rightarrow \text{colim } \mathcal{D}$. This establishes the surjectivity of $\Psi_{\mathcal{D}}$ and shows that $f : B \rightarrow C$ is of finite presentation, which is one direction of the proposition.

Assume that $\Psi_{\mathcal{D}}$ is a bijection for every directed system \mathcal{D} . Let $(\{b_i\}_{i \in I}, S)$ be a presentation of $B \rightarrow C$ such that the cardinality of $I \cup S$ is minimal. We show that both I and S are finite.

Define for every pair of finite subsets $J \subset I$ and $T \subset S$ such that all relations in T involve only elements of J the blueprint $D_{J,T} = B[b_i]_{i \in J} // \langle \mathcal{R}_T \rangle$ where \mathcal{R}_T is the pre-addition that is generated by T and \mathcal{R}_B . Then every $D_{J,T}$ is naturally a B -algebra and a pair of inclusions $J_1 \subset J_2$ and $T_1 \subset T_2$ yields a morphism $D_{J_1,T_1} \rightarrow D_{J_2,T_2}$ of B -algebras. This defines a directed system \mathcal{D} whose colimit $\text{colim } \mathcal{D}$ is C . Since $\Psi_{\mathcal{D}}$ is bijective, the identity morphism $\text{id} : C \rightarrow C = \text{colim } \mathcal{D}$ comes from an element $(\varphi_{J,T})$ of $\text{colim Hom}_B(C, \mathcal{D})$. This means that there are a finite subset J of I and a finite subset T of S such that $\text{id} : C \rightarrow C$ factorizes into a morphism $\varphi_{J,T} : C \rightarrow D_{J,T}$, followed by $\iota_{J,T} : D_{J,T} \rightarrow C$. By the minimality of I and S , this can only be the case if both $J = I$ and $T = S$, i.e. I and S are finite. This finishes the proof of the proposition. \square

6. LOCALIZATIONS

Let $B = A // \mathcal{R}$ be a blueprint. Let S be a *multiplicative set* in B , i.e. a subset of B that contains 1 and ab for all $a, b \in S$. We define $S^{-1}A$ as the quotient of $A \times S$ by the equivalence relation \sim given by $(a, s) \sim (a', s')$ if and only if there is a $t \in S$ such that $tsa' = ts'a$. We write $\frac{a}{s}$ for the equivalence class of (a, s) in $S^{-1}A$. We define $S^{-1}\mathcal{R}$ as the set

$$S^{-1}\mathcal{R} = \left\{ \sum \frac{a_i}{s_i} \equiv \sum \frac{b_j}{r_j} \mid \exists t \in S \text{ such that } \sum ts^i a_i \equiv \sum tr^j b_j \right\}$$

where

$$s^i = \prod_{k \neq i} s_k \cdot \prod_j r_j \quad \text{and} \quad r^j = \prod_i s_i \cdot \prod_{l \neq j} r_l.$$

Then $S^{-1}A$ is a monoid (with the multiplication inherited from $A \times S$) and that $S^{-1}\mathcal{R}$ is a pre-addition for $S^{-1}A$. We define the *localization of B at S* as the blueprint $S^{-1}B = S^{-1}A // S^{-1}\mathcal{R}$.

The association $a \mapsto \frac{a}{1}$ defines an epimorphism $B \rightarrow S^{-1}B$. It satisfies the universal property that every morphism $f : B \rightarrow C$ that maps S to the units of C factors uniquely through $B \rightarrow S^{-1}B$. If $S = \{h^i\}_{i \geq 0}$ is generated by some $h \in B$, then we denote $S^{-1}B$ by $B[h^{-1}]$. Note that if S is generated by finitely many elements h_1, \dots, h_n , then $S^{-1}B = B[h^{-1}]$ for $h = \prod h_i$.

Given a blue B -module M and a multiplicative subset S of B , we define $S^{-1}M$ as the following blue B -module. Its underlying set is the quotient of $M \times S$ by the equivalence relation \sim defined by $(m, s) \sim (m', s')$ if and only if there is a $t \in S$ such that $ts.m' = ts'.m$. We denote by $\frac{m}{s}$ the equivalence class of (m, s) and denote by $f : M \rightarrow S^{-1}M$ the canonical map that sends m to $\frac{m}{1}$. The pre-addition of $S^{-1}M$ is generated by $f(\mathcal{P})$ where \mathcal{P} is the pre-addition of M . With this $f : M \rightarrow S^{-1}M$ is a morphism of blue B -modules, and $S^{-1}M$ is naturally a blue $S^{-1}B$ -module.

7. BLUE SCHEMES AS RELATIVE SCHEMES

In this section, we show that every blue scheme X over a blueprint B defines a scheme h_X relative to $\mathcal{M}od B$. As a functor on $\text{Comm}(\mathcal{M}od B)$, the relative scheme h_X is defined as $\text{Hom}_B(\text{Spec}(-), X)$.

Note that in this section, we consider relative schemes as functors on $\text{Comm}(\mathcal{M}od B)$ rather than on its dual category $\text{Aff}(\mathcal{M}od B)$ to avoid cumbersome notations and possible confusions that stem from the fact that $\text{Spec} : \mathcal{B}lpr_B \rightarrow \text{Sch}_B$ is not fully faithful and therefore $\text{Aff}(\mathcal{M}od B)$ is not a full subcategory of the category of schemes relative to $\mathcal{M}od B$. For more details on this, cf. Section 9.

Lemma 6. *Let B be a blueprint and $h \in B$. Then the canonical morphism $f : B \rightarrow B[h^{-1}]$ is a flat epimorphism of finite presentation.*

Proof. Since $- \otimes_B B[h^{-1}]$ is left-adjoint to $\text{Hom}_B(B[h^{-1}], -)$ it commutes with colimits. It is easily verified that $- \otimes_B B[h^{-1}]$ commutes with finite limits (cf. [1, Prop. 2.24] for the case of a monoid B). Therefore $B \rightarrow B[h^{-1}]$ is flat.

Since the image of h^{-1} under a blueprint morphism $g : B[h^{-1}] \rightarrow C$ is determined by the image of h , $f : B \rightarrow B[h^{-1}]$ is an epimorphism.

We are left with showing that $f : B \rightarrow B[h^{-1}]$ is of finite presentation. The B -algebra $B[h^{-1}]$ is generated by h over B . Let $\tilde{f} : B[T] \rightarrow B[h^{-1}]$ be the B -algebra morphism defined by $T \mapsto h$. Let $S = \{Th \equiv 1\}$ and \mathcal{R} be the pre-addition of B . Then $(\{b\}, S)$ is a finite presentation of the B -algebra $B[h^{-1}]$, and we can employ Proposition 5 to conclude that $f : B \rightarrow B[h^{-1}]$ is of finite presentation. \square

This allows to conclude the main statement of this text. For a blue scheme X over a blueprint B , define the presheaf $h_X = \text{Hom}_B(\text{Spec}(-), X)$ on $\text{Comm}(\mathcal{M}od B)$.

Theorem 7. *The association $X \mapsto h_X$ defines a fully faithful embedding*

$$\iota : \text{Sch}_B \longrightarrow \text{Sch}(\mathcal{M}od B)$$

of categories.

Proof. A blue scheme X over B defines the presheaf $h_X = \text{Hom}_B(\text{Spec}(-), X)$ on the category $\text{Comm}(\mathcal{M}od B)$. Since the inclusion $\varphi : U \rightarrow X$ of an open subset U of X is locally defined by a localization $B \rightarrow B[h^{-1}]$, Lemma 6 implies that $\varphi : U \rightarrow X$ is a Zariski open immersion in the sense of Toën and Vaquié.

From this, the theorem follows by general arguments as they can be found in [3] for rings or in [7] for monoids in place of blueprints. \square

8. A RELATIVE SCHEME THAT IS NOT A BLUE SCHEME

The embedding $\iota : \text{Sch}_B \rightarrow \text{Sch}(\mathcal{M}od B)$ is not essential surjective. To begin with, note that the inclusions $\mathbb{F}_1 \rightarrow \mathbb{F}_{1^2}$, $\mathbb{N} \rightarrow \mathbb{Z}$ and $\mathbb{Q}_{\geq 0} \rightarrow \mathbb{Q}$ are examples of flat epimorphisms of finite presentation that are not localizations. Another type of a flat epimorphism of finite presentation is a blueprint morphism $B_1 \rightarrow B_2$ that is a bijection between the underlying monoids, but where the pre-addition of B_2 is strictly larger than and finitely generated over the pre-addition of B_1 .

The following is an example of a scheme relative to $\mathcal{M}od \mathbb{F}_1$ does not come from a blue scheme. Let $B_1 = \mathbb{F}_1[T]$, $B_2 = \mathbb{F}_1[-T]$, $B_0 = \mathbb{F}_1[\pm T]$ and $f_1 : B_1 \rightarrow B_0$ and $f_2 : B_2 \rightarrow B_0$ the obvious inclusions. Then f_1 and f_2 are flat epimorphisms of finite representation. Let $h_{U_i} = \text{Hom}(B_i, -)$ be the corresponding functor of points and define $\mathcal{X} = h_{U_1} \amalg_{h_{U_0}} h_{U_2}$, i.e. \mathcal{X} is the sheaf associated to the presheaf

$$C \mapsto \text{Hom}(\mathbb{F}_1[T], C) \amalg_{\text{Hom}(\mathbb{F}_1[\pm T], C)} \text{Hom}(\mathbb{F}_1[-T], C),$$

which glues two copies $+C$ and $-C$ of C along the subset C^\pm of elements of C with an additive inverse. Thus if C is a ring, then $\mathcal{X}(C) = C$. This means that the restriction of \mathcal{X} to rings is the affine line $\mathbb{A}_{\mathbb{Z}}^1$. For $C = \mathbb{F}_1$, $\mathcal{X}(\mathbb{F}_1)$ contains $\{0, \pm 1\} = \mathbb{F}_{1^2}$.

We lead the assumption that \mathcal{X} comes from a blue scheme X to a contradiction. If $\mathcal{X} \simeq \iota(X)$, then there is a cancellative blue scheme X_{canc} together with a canonical morphism $X_{\text{canc}} \rightarrow X$ such that every morphism from the spectrum of a cancellative blueprint factors uniquely through X_{canc} (see [4, para. 3.32]). Since \mathcal{X} equals $\mathbb{A}_{\mathbb{Z}}^1$ if restricted to rings, $(X_{\text{canc}})_{\mathbb{Z}}^+ = \mathbb{A}_{\mathbb{Z}}^1$. In particular, the morphism $U_1 = \text{Spec } \mathbb{F}_1[T] \rightarrow X$ factors uniquely through a morphism $\varphi : \text{Spec } \mathbb{F}_1[T] \rightarrow X_{\text{canc}}$, which yields the commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathbb{Z}[T] & \xrightarrow{\text{id}} & \mathbb{A}_{\mathbb{Z}}^1 \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{F}_1[T] & \xrightarrow{\varphi} & X_{\text{canc}} \end{array}$$

by base extension to rings. The vertical arrows are surjective by [5, Lemma 1.32], which implies that φ is surjective. Since h_{U_1} is a subsheaf of \mathcal{X} , φ is injective and thus a bijection. There are open affine coverings $\{V_i\}$ of $\mathbb{A}_{\mathbb{F}_1}^1 = \text{Spec } \mathbb{F}_1[T]$ and $\{W_i\}$ of X_{canc} such that φ restricts to morphisms $\varphi_i : V_i \rightarrow W_i$ (see [4, Thm. 67]). Since an open covering of $\mathbb{A}_{\mathbb{F}_1}^1$ must contain $\mathbb{A}_{\mathbb{F}_1}^1$ itself (see [5, Ex. 1.6] for a description of this space), and since φ is a bijection, X_{canc} must be affine with coordinate blueprint C and φ must come from a blueprint morphism $f = \Gamma\varphi : C \rightarrow \mathbb{F}_1[T]$. Since $C_{\mathbb{Z}}^+ = \mathbb{Z}[T]$, we conclude that $f : C \rightarrow \mathbb{F}_1[T]$ is an isomorphism of blueprints. But this leads to a contradiction because $\{0, \pm 1\} \subset \mathcal{X}(\mathbb{F}_1)$ while $\mathbb{A}_{\mathbb{F}_1}^1(\mathbb{F}_1) = \{0, 1\}$.

Note that we can replace B_0 , B_1 and B_2 by their respective universal semirings B_0^+ , B_1^+ and B_2^+ . Then the above construction yields a scheme \mathcal{X}^+ relative to modules over \mathbb{N} . It sends a semiring R to the amalgam $(+R) \amalg_{R^\pm} (-R)$, which glues two copies $+R$ and $-R$ of R along the subset R^\pm of all elements with an additive inverse. Then a similar argumentation as above shows that the relative scheme \mathcal{X}^+ does neither come from a blue scheme.

Remark 8. At the end of Section 2.4 in [6], one finds the construction of a topological space $|\mathcal{X}|$ for a relative scheme \mathcal{X} such that Zariski open immersions correspond to open subsets of $|\mathcal{X}|$. It would be interesting to compare the topological space $|\iota(X)|$ with the topological space of X .

Since $X = \text{Spec } \mathbb{F}_1$ has the non-trivial Zariski open immersion $\text{Spec } \mathbb{F}_{1^2} \rightarrow \text{Spec } \mathbb{F}_1$, the space $|\iota(X)|$ must be larger than the one point spaces $\text{Spec } \mathbb{F}_1 = \{*\}$. Similarly, the relative

scheme $\mathcal{X} = h_{U_1} \amalg_{h_{U_0}} h_{U_2}$ of this section must have a larger topological space than the topological colimit

$$\operatorname{colim}(\{\eta, x\} \xleftarrow{\operatorname{id}} \{\eta, x\} \xrightarrow{\operatorname{id}} \{\eta, x\}) = \{\eta, x\}$$

over the underlying topological spaces of U_0 , U_1 and U_2 . Indeed, the only open subsets are \emptyset , $\{\eta\}$ and $\{\eta, x\}$, which does not allow a covering by two proper open subsets $|h_{U_1}|$ and $|h_{U_2}|$.

9. A CONCLUDING REMARK ON GLOBALIZATIONS

The functor $\operatorname{Spec} : \mathcal{B}lpr \rightarrow \operatorname{Sch}_{\mathbb{F}_1}$ is not fully faithful in contrast to usual scheme theory, and the composition of Spec with the global section functor $\Gamma : \operatorname{Sch}_{\mathbb{F}_1} \rightarrow \mathcal{B}lpr$ is not isomorphic to the identity functor on $\mathcal{B}lpr$. However, the following result assures that this is not a problem for the scheme theory associated with blueprints.

Let B be a blueprint and $X = \operatorname{Spec} B$. Then we denote by ΓB the blueprint $\Gamma(X, \mathcal{O}_X)$, i.e. the image under the composition $\Gamma \circ \operatorname{Spec}$. We call ΓB the *globalization* of B . The globalization comes together with a canonical morphism $B \rightarrow \Gamma B$, which is in general not an isomorphism. If it is an isomorphism, we call B *global*. However, the associated morphism $\operatorname{Spec} \Gamma B \rightarrow \operatorname{Spec} B$ is an isomorphism of affine blue schemes (see [4, Thm. 3.12]). Therefore, $\Gamma B \rightarrow \Gamma \Gamma B$ is an isomorphism, and the category of affine blue schemes is dual to the category of global blueprints.

In Toën and Vaquié's approach, the category $\operatorname{Aff}(\mathcal{B}lpr)$ of affine relative schemes is dual to the category $\mathcal{B}lpr$ of all blueprints. In particular, an arbitrary blueprint B defines the affine relative scheme h_B that sends a blueprint C to the set $\operatorname{Hom}(B, C)$. If B is not global, then h_B is not in the essential image of $\iota : \operatorname{Sch}_{\mathbb{F}_1} \rightarrow \operatorname{Sch}(\mathcal{M}od \mathbb{F}_1)$. In particular, $\iota(\operatorname{Spec} B)$ is isomorphic to $h_{\Gamma B}$.

Another effect is that if a family of blueprint morphisms $B \rightarrow B_i$ yields an affine open covering $\{\operatorname{Spec} B_i\}$ of $\operatorname{Spec} B$, then it is in general *not* the case that the family $\{h_{B_i}\}$ is an open covering of h_B —an implication that is true if all blueprints in question are global.

Note that the globalization $B \rightarrow \Gamma B$ is always a flat epimorphism, but in general not of finite presentation. But if, for instance, B is finitely generated over \mathbb{F}_1 , then $B \rightarrow \Gamma B$ is a flat epimorphism of finite presentation and therefore $h_{\Gamma B} \rightarrow h_B$ is an open immersion of affine relative schemes for finitely generated B .

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